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Block Toeplitz operators with rational symbols

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Abstract
In this paper we show that the hyponormality of block Toeplitz operators T_Φ with matrix-valued rational symbols Φ in $L^\infty(\mathbb{C}^{n\times n})$ is completely determined by the tangential Hermite–Fejér interpolation problem.

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1. Introduction

Toeplitz operators (or equivalently, Wiener–Hopf operators; more generally, block Toeplitz operators; and particularly, Toeplitz determinants) are of importance in connection with a variety of problems in physics, and in particular, in the field of quantum mechanics. For example, a study of solvable models in quantum mechanics uses the spectral theory of Toeplitz operators (cf [Pr]); the one-dimensional Heisenberg Hamiltonian of ferromagnetism is written as a direct sum of the sums of Toeplitz operators and multiplicative potentials, so that a study on the spectral properties of Toeplitz operators is required in understanding this model (cf [DMA]); a study of quantum spin chains uses Toeplitz determinants (cf [KMN]); a study of the vicious walkers model uses the Toeplitz and Fredholm theory (cf [HI]); and the theory of block Toeplitz determinants plays an important role in the study of high-temperature superconductivity (cf [BE]). On the other hand, the theory of hyponormal operators is an extensive and highly developed area. In particular, a study of the spectral properties of hyponormal operators has made important contributions in the study of related mathematical physics problems. For example, if T is a hyponormal operator then the norm of $\|T^n\|$ can easily be computed from the n power of $\|T\|$. Also hyponormal operators enjoy Weyl's theorem, which is the statement that if $T - \lambda I$ is a non-invertible Fredholm operator of index zero then λ is an isolated eigenvalue of finite multiplicity and vice versa. Besides, hyponormal operators possess many useful spectral properties. Consequently, it is quite informative to know the hyponormality of (block) Toeplitz operators. In this paper we are concerned with the hyponormality of block Toeplitz operators with rational symbols.

A bounded linear operator A on an infinite-dimensional complex Hilbert space $\mathcal{H}$ is said to be *hyponormal* if its self-commutator $[A^*, A] = A^*A - AA^*$ is positive (semidefinite). For φ in $L^\infty(\mathbb{T})$ of the unit circle $\mathbb{T} = \partial\mathbb{D}$, the (single) Toeplitz operator with symbol φ is the operator T_φ on the Hardy space $H^2(\mathbb{T})$ defined by

$$T_\varphi f = P(\varphi f) \qquad (f \in H^2(\mathbb{T})),$$

where P denotes the orthogonal projection that maps from $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$. The problem of determining which symbols induce hyponormal Toeplitz operators was completely solved by Cowen [Co] in 1988.

Cowen's theorem ([Co], [NT])**.** *For* $\varphi \in L^\infty$, write

$$\mathcal{E}(\varphi) := \{k \in H^\infty : \|k\|_\infty \leqslant 1 \text{ and } \varphi - k\overline{\varphi} \in H^\infty\}.$$

Then T_φ *is hyponormal if and only if* $\mathcal{E}(\varphi)$ *is nonempty.*

Cowen's theorem is to recast the operator-theoretic problem of hyponormality for (single) Toeplitz operators into the problem of finding a solution to a certain functional equation involving the operator's symbol. Tractable and explicit criteria for the hyponormality of Toeplitz operators T_φ with scalar trigonometric polynomial or rational symbols φ were established by many authors (cf [Co], [CL], [HL1], [HL2], [NT], etc).

For the matrix-valued function $\Phi \in L^\infty(\mathbb{C}^{n\times n})$, the *block Toeplitz operator with symbol* Φ is the operator T_Φ on the vector-valued Hardy space $H^2(\mathbb{C}^n)$ of the unit disc defined by

$$T_\Phi h = P_n(\Phi h) \qquad (h \in H^2(\mathbb{C}^n)),$$

where P_n denotes the orthogonal projection that maps $L^2(\mathbb{C}^n)$ onto $H^2(\mathbb{C}^n)$. If we set $H^2(\mathbb{C}^n) = H^2(\mathbb{T}) \oplus \cdots \oplus H^2(\mathbb{T})$ then we see that if

$$\Phi = \begin{bmatrix} \varphi_{11} & \dots & \varphi_{1n} \\ & \vdots & \\ \varphi_{n1} & \dots & \varphi_{nn} \end{bmatrix}$$

then

$$T_\Phi = \begin{bmatrix} T_{\varphi_{11}} & \dots & T_{\varphi_{1n}} \\ & \vdots & \\ T_{\varphi_{n1}} & \dots & T_{\varphi_{nn}} \end{bmatrix}.$$

Very recently, Gu, Hendricks and Rutherford [GHR] considered the hyponormality of block Toeplitz operators and characterized the hyponormality of block Toeplitz operators in terms of their symbols. In particular, they showed that the hyponormality of the block Toeplitz operator T_Φ will force Φ to be normal, that is, $\Phi^*\Phi = \Phi\Phi^*$. Their characterization for hyponormality of block Toeplitz operators resembles the Cowen theorem with an additional condition—the normality condition of the symbol.

Theorem 1.1 [GHR]**.** *For* $\Phi \in L^\infty(\mathbb{C}^{n\times n})$, T_Φ *is hyponormal if and only if* Φ *is normal and*

$$\mathcal{E}(\Phi) := \{K \in H^\infty(\mathbb{C}^{n\times n}) : \|K\|_\infty \leqslant 1 \text{ and } \Phi - K\Phi^* \in H^\infty(\mathbb{C}^{n\times n})\}$$

is nonempty.

However the case of arbitrary matrix symbol $\Phi \in L^\infty(\mathbb{C}^{n\times n})$, though solved by theorem 1.1, is in practice very difficult because the matrix multiplication is not commutative. In [GHR] it was shown that, as in the scalar case, if $\Phi(z)$ is a trigonometric matrix polynomial with invertible leading coefficient then the hyponormality of T_Φ can be determined by a matrix-valued Caratheodory interpolation problem.

On the other hand, a function $\varphi \in L^\infty$ is said to be of *bounded type* (or in the Nevanlinna class) if there are functions ψ_1, ψ_2 in $H^\infty(\mathbb{D})$ such that

$$\varphi(z) = \frac{\psi_1(z)}{\psi_2(z)}$$

for almost all z in $\mathbb{T}$. Rational functions in L^∞ are of bounded type. For a matrix-valued function $\Phi = [\phi_{ij}] \in L^\infty(\mathbb{C}^{n\times n})$, we say that Φ is of *bounded type* if each entry ϕ_{ij} is of bounded type.

For $\Phi \in L^\infty(\mathbb{C}^{n\times n})$ write

$$\Phi_+ := P(\Phi) \in H^2(\mathbb{C}^{n\times n}) \qquad \text{and} \qquad \Phi_- := \big[(I-P)(\Phi)\big]^* \in H^2(\mathbb{C}^{n\times n}),$$

where P denotes the orthogonal projection from $L^2(\mathbb{C}^{n\times n})$ to $H^2(\mathbb{C}^{n\times n})$. Thus we can write $\Phi = \Phi_-^* + \Phi_+$. For $F = [f_{ij}] \in H^\infty(\mathbb{C}^{n\times n})$, we say that F is called *rational* if each entry f_{ij} is a rational function. Also if given $\Phi \in L^\infty(\mathbb{C}^{n\times n})$, Φ_+ and Φ_- are rational then we say that the block Toeplitz operator T_Φ has a rational symbol Φ.

In this paper we show that if $\Phi \in L^\infty(\mathbb{C}^{n\times n})$ is a rational symbol then the hyponormality of the block Toeplitz operator T_Φ can be determined by the matrix-valued tangential Hermite–Fejér interpolation problem. We here formulate the (matrix-valued) tangential Hermite–Fejér interpolation problem (cf [FF]).

Problem 1.2. *Let* $\{A_{ij} : 1 \leqslant i \leqslant N \text{ and } 0 \leqslant j < p_i\}$ *and* $\{B_{ij} : 1 \leqslant i \leqslant N \text{ and } 0 \leqslant j < p_i\}$ *be a set of* $n \times n$ *complex matrices, respectively and let* $\alpha_1, \alpha_2, \ldots, \alpha_N$ *be N distinct complex numbers in* $\mathbb{D}$. *Find necessary and sufficient conditions for the existence of a contractive analytic function K in* $H^\infty(\mathbb{C}^{n\times n})$ *satisfying that for each* $i = 1, 2, \ldots, N$,

$$\begin{bmatrix} B_{i,0} \\ B_{i,1} \\ \vdots \\ B_{i,p_i-1} \end{bmatrix} = \begin{bmatrix} K(\alpha_i) & 0 & \cdots & 0 \\ \frac{K^{(1)}(\alpha_i)}{1} & K(\alpha_i) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \frac{K^{(p_i-1)}(\alpha_i)}{(p_i-1)!} & \frac{K^{(p_i-2)}(\alpha_i)}{(p_i-2)!} & \cdots & K(\alpha_i) \end{bmatrix} \begin{bmatrix} A_{i,0} \\ A_{i,1} \\ \vdots \\ A_{i,p_i-1} \end{bmatrix}.$$

2. The main result

We begin with an observation that if $f \in H^2$ is such that $\overline{f}$ is of bounded type, say $\overline{f} = \psi_2/\psi_1$ ($\psi_1, \psi_2 \in H^\infty$) then dividing the outer part of ψ_1 into ψ_2 one obtains $\overline{f} = b/\theta$, where θ is inner and $b \in H^\infty$ satisfies that the inner parts of b and θ are coprime. Thus $f = \theta\overline{b}$. If we write, for an inner function θ,

$$\mathcal{H}(\theta) := H^2 \ominus \theta H^2,$$

then since $f \in H^2$, we must have $b \in \mathcal{H}(z\theta)$. Thus if $f \in H^2$ is such that $\overline{f}$ is of bounded type and $f(0) = 0$ then we can write (cf [Ab])

$$f = \theta\overline{b}, \tag{2.1}$$

where θ is an inner function and $b \in \mathcal{H}(\theta)$ satisfies that the inner parts of b and θ are coprime. In particular, if $f \in H^\infty$ is a rational function then f can be written as (cf [HL2])

$$f = \theta\overline{b}, \tag{2.2}$$

where θ is a finite Blaschke product and $b \in H^\infty$ satisfies that the inner parts of b and θ are coprime.

Now suppose $F = [f_{ij}] \in H^\infty(\mathbb{C}^{n\times n})$ is rational. Then in view of (2.2) we may write $f_{ij} = \theta_{ij}\overline{b_{ij}}$, where θ_{ij} is a finite Blaschke product, b_{ij} is in $H^\infty(\mathbb{C}^{n\times n})$, and θ_{ij} and the inner part of b_{ij} are coprime. Thus we can write

$$F(z) = [f_{ij}] = [\theta_{ij}\overline{b_{ij}}] = [\theta\overline{a_{ij}}] = \Theta(z)A^*(z) \qquad (\Theta = \theta I_n), \tag{2.3}$$

where $\theta := \text{LCM}(\theta_{ij})$, $A(z) \in H^\infty(\mathbb{C}^{n\times n})$ and I_n is the $n \times n$ identity matrix. Note that for each zero α of θ, the matrix $A(\alpha)$ is nonzero. In the following, when we consider the matrix-valued rational function $F \in H^\infty(\mathbb{C}^{n\times n})$, we will assume, without loss of generality, that F is of form (2.3).

We then have

Lemma 2.1. *Suppose $\Phi \equiv \Phi_-^* + \Phi_+ \in L^\infty(\mathbb{C}^{n\times n})$ is a rational symbol of the form*

$$\Phi_+ = [\theta_1\overline{a_{ij}}] = \Theta_1(z)A^*(z) \qquad \textit{and} \qquad \Phi_- = [\theta_2\overline{b_{ij}}] = \Theta_2(z)B^*(z),$$

where $\Theta_i = \theta_i I_n$ $(i = 1, 2)$ and the θ_i are finite Blaschke products. If T_Φ is hyponormal then Θ_2 divides Θ_1.

Proof. Let Φ be a rational symbol such that T_Φ is hyponormal. By theorem 1.1 there exists a matrix-valued function $K(z) \in H^\infty(\mathbb{C}^{n\times n})$ with $\|K\|_\infty \leqslant 1$ such that $\Phi_-^* - K\Phi_+^* \in H^\infty(\mathbb{C}^{n\times n})$. Observe that

$$\begin{aligned}
&\Phi_-^* - K\Phi_+^* \in H^\infty(\mathbb{C}^{n\times n})\\
&\Longleftrightarrow B(z)\Theta_2^*(z) - K(z)A(z)\Theta_1^*(z) = F(z) \quad \text{for some } F(z) \in H^\infty(\mathbb{C}^{n\times n})\\
&\Longleftrightarrow B(z)\Theta_1(z) - K(z)A(z)\Theta_2(z) = F(z)\Theta_1(z)\Theta_2(z).
\end{aligned}$$

Therefore for each zero α of θ_2, $B(\alpha)\Theta_1(\alpha) = 0$. Since $B(\alpha)$ is nonzero for each zero α of θ_2 it follows that $\Theta_1(\alpha) = 0$ and hence $\Theta_1(z) = \Theta_2(z)\Theta_0(z)$ for some finite Blaschke product Θ_0. □

We now have

Theorem 2.2. *If $\Phi \in L^\infty(\mathbb{C}^{n\times n})$ is a normal rational symbol then T_Φ is hyponormal if and only if there exists a solution to the tangential Hermite–Fejér interpolation problem 1.2, where the data are given by the symbol Φ.*

Proof. First observe that, in view of lemma 2.1, when we study the hyponormality of block Toeplitz operators with rational symbols Φ we may assume that the symbol $\Phi \equiv \Phi_-^* + \Phi_+ \in L^\infty(\mathbb{C}^{n\times n})$ is of the form

$$\Phi_+ = [\theta_1\theta_2\overline{a_{ij}}] = \Theta_1\Theta_2A^*(z) \quad \text{and} \quad \Phi_- = [\theta_1\overline{b_{ij}}] = \Theta_1B^*(z) \quad (\Theta_i := \theta_iI_n \text{ for } i = 1, 2),$$

where θ_1 and θ_2 are finite Blaschke products of degrees d_1 and d_2, respectively. Write

$$\theta_1\theta_2 = \prod_{i=1}^{N}\left(\frac{z-\alpha_i}{1-\overline{\alpha_i}z}\right)^{p_i} \qquad \text{and} \qquad \theta_1 = \prod_{i=1}^{N_1}\left(\frac{z-\alpha_i}{1-\overline{\alpha_i}z}\right)^{p_i},$$

where $d_1 = \sum_{i=1}^{N_1} p_i$ and $d_2 = \sum_{i=N_1+1}^{N} p_i$. Write

$$\mathcal{C}(\Phi) \equiv \{K \in H^\infty(\mathbb{C}^{n\times n}) : \Phi(z) - K(z)\Phi^*(z) \in H^\infty(\mathbb{C}^{n\times n})\}.$$

Observe that

$$\begin{aligned}
&K(z) \in \mathcal{C}(\Phi)\\
&\Longleftrightarrow \Theta_1^*(z)B(z) - \Theta_2^*(z)\Theta_1^*(z)K(z)A(z) = F(z) \quad \text{for some} \quad F(z) \in H^\infty(\mathbb{C}^{n\times n})\\
&\Longleftrightarrow \Theta_2(z)B(z) - K(z)A(z) = \Theta_1(z)\Theta_2(z)F(z).
\end{aligned} \tag{2.4}$$

Put

$$A_{i,j} := \frac{A^{(j)}(\alpha_i)}{j!} \qquad \text{and} \qquad B_{i,j} := \frac{(\Theta_2 B)^{(j)}(\alpha_i)}{j!} \qquad (1 \leqslant i \leqslant N, 0 \leqslant j < p_i).$$

Then the last equation of (2.4) holds if and only if the following equations hold: for each $i = 1, \ldots, N$,

$$\begin{bmatrix} B_{i,0} \\ B_{i,1} \\ \vdots \\ B_{i,p_i-1} \end{bmatrix} = \begin{bmatrix} K(\alpha_i) & 0 & \cdots & 0 \\ \frac{K^{(1)}(\alpha_i)}{1} & K(\alpha_i) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \frac{K^{(p_i-1)}(\alpha_i)}{(p_i-1)!} & \frac{K^{(p_i-2)}(\alpha_i)}{(p_i-2)!} & \cdots & K(\alpha_i) \end{bmatrix} \begin{bmatrix} A_{i,0} \\ A_{i,1} \\ \vdots \\ A_{i,p_i-1} \end{bmatrix}. \tag{2.5}$$

Thus K is in $\mathcal{C}(\Phi)$ if and only if K is a function in $H^\infty(\mathbb{C}^{n\times n})$ satisfying (2.5). If in addition $\|K\|_\infty \leqslant 1$ is required then this is exactly the tangential Hermite–Fejér interpolation problem. Consequently, T_Φ is hyponormal if and only if the tangential Hermite–Fejér interpolation problem is solvable. This completes the proof. □

To obtain a concrete solution of theorem 2.2, we recall the solution of the tangential Hermite–Fejér interpolation problem (cf [FF]). Put

$$\widehat{A}_i = \begin{bmatrix} A_{i,0} & 0 & \cdots & 0 \\ A_{i,1} & A_{i,0} & \cdots & 0 \\ \vdots & \vdots & & 0 \\ A_{i,p_i-1} & A_{i,p_i-2} & \cdots & A_{i,0} \end{bmatrix} \qquad \text{and} \qquad \widehat{B}_i = \begin{bmatrix} B_{i,0} & 0 & \cdots & 0 \\ B_{i,1} & B_{i,0} & \cdots & 0 \\ \vdots & \vdots & & 0 \\ B_{i,p_i-1} & B_{i,p_i-2} & \cdots & B_{i,0} \end{bmatrix}$$

$$(i = 1, \ldots, N).$$

Let $\widehat{A}$ and $\widehat{B}$ be the matrices on $l^2_d(\mathbb{C}^N)$ defined by

$$\widehat{A} := \operatorname{diag}[\widehat{A}_1, \widehat{A}_2, \ldots, \widehat{A}_N] \qquad \text{and} \qquad \widehat{B} := \operatorname{diag}[\widehat{B}_1, \widehat{B}_2, \ldots, \widehat{B}_N].$$

Note that $\widehat{B}_i = 0$ for $i = N_1 + 1, \ldots, N_0$. Put

$$x_i^j := \frac{z^j}{(1-\alpha_i z)^{j+1}} \qquad (1 \leqslant i \leqslant n, 0 \leqslant j < p_i)$$

and define

$$G := \begin{bmatrix} \overline{\langle x_1^{p_1-1}, x_1^{p_1-1}\rangle} I & \cdots & \overline{\langle x_1^{p_1-1}, x_1^{0}\rangle} I & \overline{\langle x_1^{p_1-1}, x_2^{p_2-1}\rangle} I & \cdots & \overline{\langle x_1^{p_1-1}, x_2^{0}\rangle} I & \cdots & \overline{\langle x_1^{p_1-1}, x_n^{0}\rangle} I \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \overline{\langle x_n^{0}, x_1^{p_1-1}\rangle} I & \cdots & \overline{\langle x_n^{0}, x_1^{0}\rangle} I & \overline{\langle x_n^{0}, x_2^{p_2-1}\rangle} I & \cdots & \overline{\langle x_n^{0}, x_2^{0}\rangle} I & \cdots & \overline{\langle x_n^{0}, x_n^{0}\rangle} I \end{bmatrix}.$$

Note that

$$\langle x_i^j, x_k^r\rangle = \frac{\overline{x_i^{j\,(r)}(\alpha_k)}}{r!}.$$

Recall [[FF], theorem 4.3] that there exists a solution to the tangential Hermite–Fejér interpolation problem (2.5) if and only if $\widehat{A}^* G \widehat{A} - \widehat{B}^* G \widehat{B}$ is positive semidefinite.

As a consequence of the above argument we get

Corollary 2.3. *Let $\Phi \equiv \Phi_-^* + \Phi_+ \in L^\infty(\mathbb{C}^{n\times n})$ be a normal rational symbol of the form*

$$\Phi_+ = \Theta_1\Theta_2 A^*(z) \quad \text{and} \quad \Phi_- = \Theta_1 B^*(z) \quad (\Theta_i := \theta_i I_n \text{ for } i = 1, 2),$$

where θ_1 and θ_2 are finite Blaschke products. Then

$$T_\Phi \text{ is hyponormal} \iff \widehat{A}^* G \widehat{A} - \widehat{B}^* G \widehat{B} \geqslant 0.$$

We conclude with a revealing example.

Example 2.4. Let $b(z) = \frac{z-\frac{1}{2}}{1-\frac{1}{2}z}$ and let

$$T_\Phi \equiv \begin{bmatrix} T_b^* + \alpha T_b & T_z \\ T_z^* & T_b^* + \alpha T_b \end{bmatrix} \qquad (\alpha \in \mathbb{R}).$$

Then T_Φ is hyponormal if and only if $\alpha = 1$.

Proof. We use the criterion of corollary 2.3. Observe that

$$\Phi = \begin{bmatrix} \overline{b(z)} + \alpha b(z) & z \\ \overline{z} & \overline{b(z)} + \alpha b(z) \end{bmatrix}.$$

Thus Φ is a normal rational symbol. We also have

$$\Phi_+ = zb(z) \begin{bmatrix} \alpha z & 0 \\ b(z) & \alpha z \end{bmatrix}^* \qquad \text{and} \qquad \Phi_- = zb(z) \begin{bmatrix} z & 0 \\ b(z) & z \end{bmatrix}^*,$$

and hence

$$\widehat{A}_{1,0} = A(\alpha_1) = A(0) = \begin{bmatrix} 0 & 0 \\ -\frac{1}{2} & 0 \end{bmatrix}, \qquad \widehat{A}_{2,0} = A(\alpha_2) = A\left(\frac{1}{2}\right) = \begin{bmatrix} \frac{1}{2}\alpha & 0 \\ 0 & \frac{1}{2}\alpha \end{bmatrix},$$

$$\widehat{B}_{1,0} = B(\alpha_1) = B(0) = \begin{bmatrix} 0 & 0 \\ -\frac{1}{2} & 0 \end{bmatrix}, \qquad \text{and} \qquad \widehat{B}_{2,0} = B(\alpha_2) = B\left(\frac{1}{2}\right) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

We thus have

$$\widehat{A} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\alpha & 0 \\ 0 & 0 & 0 & \frac{1}{2}\alpha \end{bmatrix}, \qquad \widehat{B} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}, \qquad G = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & \frac{4}{3} & 0 \\ 0 & 1 & 0 & \frac{4}{3} \end{bmatrix},$$

and

$$\widehat{A}^* G \widehat{A} - \widehat{B}^* G \widehat{B} = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{4}\alpha + \frac{1}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3}\alpha^2 - \frac{1}{3} & 0 \\ -\frac{1}{4}\alpha + \frac{1}{4} & 0 & 0 & \frac{1}{3}\alpha^2 - \frac{1}{3} \end{bmatrix}.$$

A straightforward calculation shows that $\widehat{A}^* G \widehat{A} - \widehat{B}^* G \widehat{B}$ is positive if and only if $\alpha = 1$. □

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